# Corrected entropy of the rotating black hole solution of the new massive gravity using the tunneling method and Cardy formula

Behrouz Mirza* and Zeinab Sherkatghanad†
*Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran*

We study the AdS rotating black hole solution for the Bergshoeff-Hohm-Townsend (BHT) massive gravity in three dimensions. The field equations of the asymptotically AdS black hole of the static metric can be expressed as the first law of thermodynamics, i.e. $dE = TdS - PdV$. The corrected Hawking-like temperature and entropy of asymptotically AdS rotating black hole are calculated using the Cardy formula and the tunneling method. Comparison of these methods will help identify the unknown leading correction parameter $\beta_1$ in the tunneling method.

PACS numbers: 04.60.Kz, 04.70.Dy



## I. INTRODUCTION

As shown by Hawking, black holes are not really black but emit all kinds of particles with a perfect black body spectrum [1–4]. The Hawking black hole temperature is proportional to its surface gravity and its entropy is proportional to its horizon area. This is known as the celebrated Bekenstein-Hawking area law $S_{BH} = \frac{A}{4}$. There are several different methods for calculating the correction to the semiclassical Bekenstein-Hawking entropy. These are based on statistical mechanical arguments, field theory methods, quantum geometry, Cardy formula, tunneling method, etc. (The corresponding literature is rather extensive; for a partial selection, see Refs. [5–10]). Among these, the tunneling method is based on two variants called null geodesic method [11] and Hamiltonian-Jacobi method [12].

Bergshoeff, Hohm and Townsend (BHT) recently advanced a theory of massive gravity in three dimensions with remarkable properties [13]. The BHT theory appears to be unitary and renormalizable and several exact solutions have been found [14, 15]. The theory is described by the parity-invariant action:

$$I_{BHT} = \frac{1}{16\pi G}\int d^3x\sqrt{-g}[R - 2\lambda - \frac{1}{m^2}(R_{\mu\nu}R^{\mu\nu} - \frac{3}{8}R^2)] \quad (1)$$

This action yields fourth order field equations for the metric. It is known that, for a special case where $m^2 = \lambda$, the theory has a unique maximal solution with interesting features such as enhancement of gauge invariance for the linearized theory. It is also shown that, in this same case, the Brown-Henneaux boundary conditions can be consistently relaxed, which enlarges the space of admissible solutions so as to include rotating black holes, gravitational solitons, kinks and wormholes [16, 17].

Some explicit expressions have been computed for the corrected entropy of such black holes as Lovelock and BTZ [19–23]. In this paper, we calculate the corrected temperature and entropy for an asymptotically AdS rotating black hole solution of the new massive gravity [24].

*Electronic address: b.mirza@cc.iut.ac.ir
†Electronic address: z.sherkat@ph.iut.ac.ir

The outline of this paper is as follows: In Section 2, we will consider the field equations of the black hole solution for the new massive gravity theory at the horizons express in the first law of thermodynamics. In Section 3, the thermodynamic properties of a rotating black hole solution are summarized and the semiclassical entropy and temperature are written. In Section 4, the Cardy formula is used to compute the first-order quantum correction to the Bekenstein-Hawking entropy. In Sections 5 and 6, the corrected Hawking temperature and entropy are calculated using the tunneling method. In Section 7, an unknown parameter $\beta_1$ in the tunneling method is identified.

## II. RELATIONSHIP BETWEEN FIELD EQUATIONS AND THERMODYNAMICS

In the special case of $m^2 = \lambda$ and negative cosmological constant, $\lambda = -\frac{1}{2l^2}$, the metric of the BHT massive gravity theory admits the following exact solution [17]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\phi^2, \quad (2)$$

where,

$$f(r) = \frac{r^2}{l^2} + br - \mu, \quad (3)$$

$$b = -\frac{1}{l^2}(r_+ + r_-), \quad (4)$$
$$\mu = -\frac{r_+ r_-}{l^2}.$$

Now, let us consider $L_m$, Lagrangian density of matter fields, in the action of the BHT massive gravity theory. Varying the action yields corresponding equation of motion

$$G_{\mu\nu} + \lambda g_{\mu\nu} - \frac{1}{2m^2}K_{\mu\nu} = 8\pi G T^m_{\mu\nu}, \quad (5)$$

where,

$$\begin{aligned} K_{\mu\nu} &\equiv 2\nabla^2 R_{\mu\nu} - \frac{1}{2}(\nabla_\mu\nabla_\nu R + g_{\mu\nu}\nabla^2 R) \\ &- 8R_{\mu\rho}R^\rho_\nu + \frac{9}{2}RR_{\mu\nu} + g_{\mu\nu}[3R^{\alpha\beta}R_{\alpha\beta} - \frac{13}{8}R^2], \end{aligned} \quad (6)$$

Here, $T^m_{\mu\nu}$ is the energy-momentum tensor for matter fields. The components of the Einstein tensor at the horizons for the metric (2) are given by [18]

$$G^0_0|_{r=r_+} = G^1_1|_{r=r_+} = \frac{f'(r_+)}{2r_+} \quad (7)$$
$$G^0_0|_{r=r_-} = G^1_1|_{r=r_-} = \frac{f'(r_-)}{2r_-}.$$

Also, the field equations (5) can be cast in the following form

$$G_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi G(T^m_{\mu\nu} + \frac{1}{8\pi G}T^{(cur)}_{\mu\nu}), \quad (8)$$

where,

$$T^{(cur)}_{\mu\nu} = \frac{1}{2m^2}K_{\mu\nu}. \quad (9)$$

It is straightforward to show that $T^{0(cur)}_0$ and $T^{1(cur)}_1$ are the same at the horizons: $r_+$ and $r_-$.

$$T^{0(cur)}_0|_{r=r_+} = T^{1(cur)}_1|_{r=r_+} = \frac{b}{2r_+} + \frac{1}{2l^2}$$
$$T^{0(cur)}_0|_{r=r_-} = T^{1(cur)}_1|_{r=r_-} = \frac{b}{2r_-} + \frac{1}{2l^2}, \quad (10)$$

From the above equations and the field equations for the stress-energy tensor of matter fields at the horizons, we have

$$T^{0(m)}_0 = T^{1(m)}_1. \quad (11)$$

At the horizons, the 0-0 component of the field equations can be written as

$$\frac{f'(r_+)}{2r_+} - \frac{1}{l^2} - \frac{b}{2r_+} = 8\pi G P_{r_+} \quad (12)$$

and

$$\frac{f'(r_-)}{2r_-} - \frac{1}{l^2} - \frac{b}{2r_-} = 8\pi G P_{r_-}, \quad (13)$$

where, $P_r$ is the radial pressure. Consider a displacement $dr_+$ and $dr_-$ multiplied by both sides of (12) and (13) when they are subtracted $[f'(r_+) = -f'(r_-)]$, we get

$$\begin{aligned}\frac{f'(r_+)}{4\pi}(\frac{\pi}{2G}d(r_+ - r_-)) &- \frac{r_+ - r_-}{8Gl^2}(dr_+ - dr_-) \\ &= Pd(V_+ - V_-). \quad (14)\end{aligned}$$

The Hawking temperature and Bekenstein-Hawking entropy are given by

$$T = \frac{r_+ - r_-}{4\pi l^2} = \frac{f'(r_+)}{4\pi} \quad (15)$$
$$S = \frac{A_+ - A_-}{4G}, \quad (16)$$

where, $A_\pm = 2\pi r_\pm$. Thus, the first term in Eq (14) represents $TdS$ and the second term can be identified as the mass term. Finally, this equation can be written as follows

$$TdS - d(\Delta M) = Pd(V_+ - V_-) \quad (17)$$

which is same as the first law of thermodynamics.

## III. THERMODYNAMICS OF A ROTATING BLACK HOLE

Considering the special case ($m^2 = \lambda = -\frac{1}{2l^2}$) for the BHT theory, the rotating black hole solution is given in the following way [16, 17]

$$ds^2 = -NFdt^2 + \frac{dr^2}{F} + r^2(d\phi + N^\phi dt)^2, \quad (18)$$

here

$$\begin{aligned}N &= [1 + \frac{bl^2}{4H}(1 - \Xi^{\frac{1}{2}})]^2 \\ N_\Phi &= -\frac{a}{2r^2}(4GM - bH) \\ F &= \frac{H^2}{r^2}[\frac{H^2}{l^2} + \frac{b}{2}(1 + \Xi^{\frac{1}{2}})H + \frac{b^2l^2}{16}(1 - \Xi^{\frac{1}{2}})^2 - 4GM\Xi^{\frac{1}{2}}],\end{aligned} \quad (19)$$

and

$$H = [r^2 - 2GMl^2(1 - \Xi^{\frac{1}{2}}) - \frac{b^2l^4}{16}(1 - \Xi^{\frac{1}{2}})^2]^{\frac{1}{2}}, \quad (20)$$

where $\Xi = 1 - \frac{a^2}{l^2}$, the parameter $a$ is bounded by AdS radius $-l \le a \le l$ . This solution is explained by two global charges, being $M$ and $J = Ma$ which are the mass and angular momentum, respectively, as well as by an additional "gravitational hair" parameter, $b$.

The event horizon radius and temperature can be expressed in a more convenient form $r_+ = \gamma\bar{r}_+$ and $T = \gamma^{-1}\bar{T}$ where $\gamma^2 = \frac{1}{2}(1 + \Xi^{-\frac{1}{2}})$. Here, $\bar{r}_+$ is identified as the event horizon radius and $\bar{T}$ as the temperature for the static case. Angular velocity of the horizon is given by [24]

$$\Omega_+ = \frac{1}{a}(\Xi^{\frac{1}{2}} - 1). \quad (21)$$

The semi-classical Hawking temperature can be derived as follows

$$T = \frac{\hbar}{\pi l}\Xi^{\frac{1}{2}}\sqrt{2G\Delta M(1 + \Xi^{\frac{1}{2}})^{-1}}, \quad (22)$$

where

$$\Delta M := M - M_0 = M + \frac{b^2l^2}{16G}. \quad (23)$$

Consider the first law of thermodynamics for a chargeless rotating black hole

$$d(\Delta M) = TdS - \Omega_+ d(\Delta J), \quad (24)$$

here $\Delta J = Ma - M_0 a$. We can express $dS$ as a first order differential equation

$$\begin{aligned}dS &= \frac{d(\Delta M)}{T} + \frac{\Omega_+}{T}d(\Delta J) \\ &= (\frac{1}{T} + \frac{a\Omega_+}{T})dM - (\frac{1}{T} + \frac{a\Omega_+}{T})dM_0 + \frac{\Omega_+(M - M_0)}{T}da,\end{aligned} \quad (25)$$

The angular velocity (21) can be simplified to

$$\Omega_+ = -\frac{\Delta J}{l^2(\Delta M + \sqrt{\Delta M^2 - \frac{\Delta J^2}{l^2}})}. \quad (26)$$

First order partial differential equation (25) is exact if it fulfills the following integrability conditions [10]

$$\begin{aligned}
\frac{\partial}{\partial M}(\frac{1}{T} + \frac{a\Omega_+}{T}) &= -\frac{\partial}{\partial M_0}(\frac{1}{T} + \frac{a\Omega_+}{T}) \\
\frac{\partial}{\partial a}(\frac{1}{T} + \frac{a\Omega_+}{T}) &= \frac{\partial}{\partial M}(\frac{\Omega_+(M-M_0)}{T}) \\
\frac{\partial}{\partial a}(\frac{1}{T} + \frac{a\Omega_+}{T}) &= -\frac{\partial}{\partial M_0}(\frac{\Omega_+(M-M_0)}{T}). \quad (27)
\end{aligned}$$

Using these three conditions, the solution of (25) is given by

$$S = \frac{\pi l}{\hbar}\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})}, \quad (28)$$

For $b = 0$, the entropy and temperature in these solutions reduce to the BTZ black hole [19, 23].

## IV. CORRECTION TO THE SEMICLASSICAL ENTROPY USING THE CARDY FORMULA

The Bekenstein-Hawking entropy of a black hole may be computed by counting microscopic states and using the Cardy formula [7]. The density of states could be determined by contour integration from the partition function. The partition function is given by

$$Z(\tau,\bar{\tau}) = \mathrm{Tr} e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0} = \sum \rho(\Delta,\bar{\Delta}) e^{2\pi i\Delta\tau} e^{-2\pi i\bar{\Delta}\bar{\tau}}. \quad (29)$$

For a unitary theory, $\rho$ is the number of states with eigenvalues $L_0 = \Delta$ and $\bar{L}_0 = \Delta$, as witnessed by inserting a complete set of states into the trace. For a nonunitary theory, $\rho$ is the difference between the number of positive- and negative-norm states with appropriate eigenvalues. Consider $q = e^{2\pi i\tau}$ and $\bar{q} = e^{2\pi i\bar{\tau}}$, we get

$$\rho(\Delta,\bar{\Delta}) = \frac{1}{(2\pi i)^2}\int \frac{dq}{q^{\Delta+1}}\frac{d\bar{q}}{\bar{q}^{\bar{\Delta}+1}} Z(q,\bar{q}), \quad (30)$$

One can easily find $\rho(\Delta)$ in this way

$$\rho(\Delta) \approx (\frac{c}{96\Delta^3})^{\frac{1}{4}} \exp 2\pi\sqrt{\frac{c\Delta}{6}}. \quad (31)$$

As shown in [17, 24], the algebra of the conserved charges also acquires a central extension being twice the value found for general relativity

$$c = \bar{c} = \frac{3l}{G}. \quad (32)$$

Now, the generators of the Brown-Henneaux Virasoro algebras can be computed explicitly, simply as the Hamiltonian and momentum constraints of general relativity smeared against appropriate vector fields, for the rotating black hole

$$\begin{aligned}
\Delta &= \frac{1}{2}(l\Delta M + \Delta J) = \frac{1}{2}\Delta M(l+a) \quad (33) \\
\bar{\Delta} &= \frac{1}{2}(l\Delta M - \Delta J) = \frac{1}{2}\Delta M(l-a).
\end{aligned}$$

Using (32) and (33), we have

$$2\pi\sqrt{\frac{c\Delta}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{\Delta}}{6}} = \pi l\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})}, \quad (34)$$

which is exactly equal to the semiclassical entropy. To obtain the logarithmic correction to the entropy, we calculate the density of states from (31)

$$\begin{aligned}
\rho(\Delta,\bar{\Delta}) &\approx (\frac{l}{4G((l\Delta M)^2 - (\Delta J)^2)^{\frac{3}{2}}})^{\frac{1}{2}} \exp(\pi l\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})}) \\
&\approx (\frac{l}{4Gl^3(\Delta M)^3(1-\frac{a^2}{l^2})^{\frac{3}{2}}})^{\frac{1}{2}} \exp(\pi l\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})}).
\end{aligned} \quad (35)$$

Hence, the corrected entropy is given by

$$\begin{aligned}
S &\sim \pi l\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})} \\
&- \frac{3}{2}\ln(\pi l\sqrt{\frac{2}{G}\Delta M(1+\Xi^{\frac{1}{2}})}) - \frac{3}{2}\ln \kappa l + const, \quad (36)
\end{aligned}$$

where,

$$\kappa = \frac{2}{l}\Xi^{\frac{1}{2}}\sqrt{\frac{2G\Delta M}{1+\Xi^{\frac{1}{2}}}} \quad (37)$$

is the surface gravity.

## V. CORRECTIONS TO THE SEMICLASSICAL HAWKING TEMPERATURE

In this section, we consider the scalar particle by using tunneling method beyond semiclassical approximation for finding the correction to the semiclassical Hawking temperature for an asymptotically AdS rotating black hole. For this purpose, we should isolate the $r-t$ sector of the metric from the angular part. A coordinate transformation near the horizon approximation can be found

$$d\chi = d\phi + \Omega_+ dt, \quad (38)$$

Also the metric (19) takes the form

$$ds^2 = -NF dt^2 + \frac{dr^2}{F} + r_+^2 d\chi^2, \quad (39)$$

where, the $r-t$ sector is isolated from the angular part. The massless particle in this spacetime obeys the following Klein-Gordon equation

$$-\frac{\hbar^2}{\sqrt{-g}}\partial_\mu[g^{\mu\nu}\sqrt{-g}\partial_\nu]\phi = 0. \quad (40)$$

To solve Eq (40) with the background metric (39), we can write the standard WKB anstaz for $\phi$ as

$$\phi(r,t) = exp[-\frac{i}{\hbar}S(r,t)], \tag{41}$$

Substituting the relation (41) in (40), we get

$$\frac{1}{NF}(\frac{\partial S}{\partial t})^2 \; - \; F(\frac{\partial S}{\partial r})^2 - \frac{\hbar}{i}\frac{1}{NF}(\frac{\partial^2 S}{\partial t^2}) + \frac{\hbar}{i}F(\frac{\partial^2 S}{\partial r^2}) + \; \frac{\hbar}{i}\partial_r F(\frac{\partial S}{\partial t}) + \frac{\hbar}{i}\frac{\partial_r\sqrt{N}}{\sqrt{N}}F(\frac{\partial S}{\partial r}) = 0, \tag{42}$$

In this case, $S(r,t)$ can be expanded in power of $\hbar$

$$S(r,t) = S_0(r,t) + \sum_i \hbar^i S_i(r,t). \tag{43}$$

Putting $S(r,t)$ from (43) in (42) will yield the following set of equations

$$\begin{aligned} \hbar^0 &: \frac{\partial S_0}{\partial t} = \pm\sqrt{N}F\frac{\partial S_0}{\partial r} \qquad (44) \\ \hbar^1 &: \frac{\partial S_1}{\partial t} = \pm\sqrt{N}F\frac{\partial S_1}{\partial r} \\ \hbar^2 &: \frac{\partial S_2}{\partial t} = \pm\sqrt{N}F\frac{\partial S_2}{\partial r} \\ &. \\ &. \\ &. \end{aligned}$$

All the equations are identical and their solutions are related to each other. We assume that any $S_i(r,t)$ can differ from $S_0(r,t)$ by a proportionality factor. The general form of the action is given by

$$S(r,t) = (1 + \sum_i \gamma_i \hbar^i)S_0(r,t), \tag{45}$$

The dimension of $\gamma_i$ is equal to the dimension of $\hbar^{-i}$; thus, $\gamma_i$ can be expressed in terms of dimensionless constant. In $(2+1)$ dimensions, $\hbar$ can be replaced by Planck length$(l_P)$, the length parameters for these black holes are $r_+$ and $r_-$. We have

$$S(r,t) = (1 + \sum_i \frac{\beta_i \hbar^i}{(a_1 r_+ + a_2 r_-)^i})S_0(r,t), \tag{46}$$

where $\beta_i$'s are dimensionless constants and $a_1$ and $a_2$ will be calculated in a later stage. We can isolate the semiclassical action for the $r-t$ sector near the horizon

$$S_0(r,t) = \omega t + \tilde{S}_0(r), \tag{47}$$

The total energy of the tunneling particle near the horizon approximation is given by

$$\omega = E - J\Omega_+. \tag{48}$$

Substituting (47) in the first equation of (44) yields,

$$\tilde{S}_0 = \pm\omega\int\frac{dr}{\sqrt{N}F}, \tag{49}$$

the $+(-)$ sign shows that the particle is ingoing (outgoing). As a result of substituting $S_0(r,t)$ from (47) and $\tilde{S}_0$ from (49), one is able to find (46) as follows

$$S(r,t) = (1 + \sum_i \frac{\beta_i \hbar^i}{(a_1 r_+ + a_2 r_-)^i})(\omega\, t \pm \omega\int\frac{dr}{\sqrt{N}F}). \tag{50}$$

A solution for the scalar field in the presence of the higher order corrections to the semiclassical action is given by

$$\phi_{in} = \exp[-\frac{i}{\hbar}(1 + \sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})(\omega\, t + \omega\int\frac{dr}{\sqrt{N}F})] \tag{51}$$

$$\phi_{out} = \exp[-\frac{i}{\hbar}(1 + \sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})(\omega\, t - \omega\int\frac{dr}{\sqrt{N}F})]. \tag{52}$$

where ingoing and outgoing particles cross the event horizon on different paths. Since the metric coefficients for $r-t$ sector change sign at the two sides of the event horizon, the outgoing particle cannot cross the event horizon classically. Therefore, the path on which tunneling takes place has an imaginary time coordinate (Im$t$). We can write the ingoing and outgoing probabilities as

$$P_{in} = |\phi_{in}|^2 = \exp[\frac{2}{\hbar}(1+\sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})(\omega \text{Im}\, t + \omega \text{Im} \int \frac{dr}{\sqrt{NF}})], \tag{53}$$

$$P_{out} = |\phi_{out}|^2 = \exp[\frac{2}{\hbar}(1+\sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})(\omega \text{Im}\, t - \omega \text{Im} \int \frac{dr}{\sqrt{NF}})]. \tag{54}$$

In the classical limit, the ingoing particle probability is unity; therefore, we have

$$\text{Im}\, t = -\text{Im} \int \frac{dr}{\sqrt{NF}}, \tag{55}$$

Now $P_{out}$ can be expressed as

$$P_{out} = \exp[-\frac{4}{\hbar}\omega(1+\sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})\text{Im} \int \frac{dr}{\sqrt{NF}}], \tag{56}$$

We identify the temperature of this black hole by using the principle of detailed balance for the ingoing and outgoing probabilities

$$\frac{P_{out}}{P_{in}} = \exp(-\frac{\omega}{T_H}). \tag{57}$$

The paths for the ingoing and outgoing particles crossing the event horizon are not the same and the outgoing particle cannot cross the event horizon classically. Considering the ingoing probability to be unitary, we can write

$$P_{out} = \exp(-\frac{\omega}{T_H}). \tag{58}$$

As a result, the corrected Hawking temperature becomes

$$T_H = T(1+\sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})^{-1}, \tag{59}$$

where, $T$ is the semiclassical Hawking temperature and other terms are corrections to the higher order quantum effects.

## VI. ENTROPY CORRECTION

Here, we are able to calculate the corrected entropy by using the expression of the modified temperature computed in (59)

$$T_H = T(1+\sum_i \beta_i \frac{\hbar^i}{(a_1 r_+ + a_2 r_-)^i})^{-1}. \tag{60}$$

Including modified Hawking temperature in the above equation, we find a similar expression for the thermodynamics law (25)

$$\begin{aligned} dS_{bh} &= \frac{d(\Delta M)}{T_H} + \frac{\Omega_+}{T_H} d(\Delta J) \\ &= (\frac{1}{T_H} + \frac{a\Omega_+}{T_H})dM - (\frac{1}{T_H} + \frac{a\Omega_+}{T_H})dM_0 + \frac{\Omega_+(M-M_0)}{T_H} da, \end{aligned} \tag{61}$$

Eq. (61) is an exact first order differential equation if it fulfills the following integrability conditions

$$\begin{aligned} \frac{\partial}{\partial M}(\frac{1}{T_H} + \frac{a\Omega_+}{T_H}) &= -\frac{\partial}{\partial M_0}(\frac{1}{T_H} + \frac{a\Omega_+}{T_H}) \\ \frac{\partial}{\partial a}(\frac{1}{T_H} + \frac{a\Omega_+}{T_H}) &= \frac{\partial}{\partial M}(\frac{\Omega_+(M-M_0)}{T_H}) \\ \frac{\partial}{\partial a}(\frac{1}{T_H} + \frac{a\Omega_+}{T_H}) &= -\frac{\partial}{\partial M_0}(\frac{\Omega_+(M-M_0)}{T_H}), \end{aligned} \tag{62}$$

By substituting $\frac{\Omega_+}{T_H}$ and $\frac{1}{T_H}$ from their definitions from (26) and (60) in (62), we find $a_1 = 1, a_2 = -1$. Now, we are able to continue the procedure applied for the semiclassical entropy by setting up a new dictionary and using the functional forms in which all semiclassical quantities are replaced by their corrected forms

$$S_{bh} = \int (\frac{1}{T} + \frac{a\Omega_+}{T}) \sum_i (1+\beta_i \frac{\hbar^i}{(r_+ - r_-)^i}) dM. \tag{63}$$

The quantum corrections up to the second order corrections in Eq (63) can be written as

$$\begin{aligned} S_{bh} = \int (\frac{1}{T} + \frac{a\Omega_+}{T})[1+\beta_1 \frac{\hbar}{(r_+ - r_-)} + \beta_2 \frac{\hbar^2}{(r_+ - r_-)^2} \\ +O(\hbar^3)]\, dM. \end{aligned} \tag{64}$$

For the corrected entropy, we get

$$\begin{aligned} S_{bh} &= \frac{\pi l}{\hbar}\sqrt{\frac{2}{G}(\Delta M + \sqrt{\Delta M^2 - \frac{\Delta J^2}{l^2}})} \\ &+ \beta_1 \frac{\pi}{2G} \log[\frac{\pi l}{\hbar}\sqrt{\frac{2}{G}(\Delta M + \sqrt{\Delta M^2 - \frac{\Delta J^2}{l^2}})}] \\ &- \beta_2 \frac{\pi^2}{4G^2} \frac{\hbar}{\pi l \sqrt{\frac{2}{G}(\Delta M + \sqrt{\Delta M^2 - \frac{\Delta J^2}{l^2}})}} \\ &+ O(\hbar^3). \end{aligned} \tag{65}$$

Thus the corrected entropy can be written in terms of semiclassical entropy

$$S_{bh} = S_{cl} + \beta_1 \frac{\pi}{2G}\log(S_{cl}) - \beta_2 \frac{\pi^2}{4G^2}\frac{1}{S_{cl}} + O(\hbar^3). \tag{66}$$

In this equation, the first sentence is the semiclassical entropy that, in the limit $b = 0$, is equal to the entropy of BTZ black hole. The other terms are related to the quantum corrections.

A standard formalism to identify the first coefficient of the leading correction is trace anomaly [20–22]. However, by comparing (36) and (67), we have $\beta_1 = -\frac{3G}{\pi}$.

It should be noted that the semiclassical entropy $S_{cl}$ can be calculated by Wald's formula [17, 25]. Wald's formula calculates the entropy of the black hole when the action contains terms of higher order in the curvature tensor. This formula mentions the entropy as an integral over the horizon of the black hole in the following way,

$$S = -2\pi \int_{\Sigma_h} \frac{\partial L}{\partial R_{\mu\nu\rho\lambda}} \varepsilon_{\mu\nu}\varepsilon_{\rho\lambda}\bar{\varepsilon}. \tag{67}$$

where $L$ is the lagrangian, $\varepsilon_{\mu\nu}$ is the binormal vector to the space-like bifurcation surface $\Sigma_h$ and $\bar{\varepsilon}$ represents the volume form [29]. However, it is usually not simple to relate Wald's formula to the logarithmic correction that emerges from different theories related to quantum gravity such as loop gravity, entanglement entropy, tunneling method etc [26, 27]. For obtaining quantum corrections to the entropy by Wald's formula we need to know full effective action. Tunneling method provides us with a corrected entropy and temperature which can also be applied to black holes with the higher derivative terms such as Lovelock black holes [21]. We do not know a systematic method to produce higher derivative terms in the Lagrangian and then use Wald's formula to get entropy corrections. This is an interesting problem and its solution may sheds light on the universality problem that exist in different theories of quantum gravity [28].

## VII. CONCLUSION

We express the field equations of AdS black hole solution of the new massive gravity as the first law of thermodynamics. The corrected semi-classical entropy of the asymptotically AdS rotating black hole of the new massive gravity is calculated by using the cardy formula and tunneling formalism. It is found that the leading correction to the semiclassical entropy for black hole is logarithmic and next to the leading correction is the inverse of the semiclassical entropy. The two methods yield the same results. The leading correction coefficient can be obtained by using the Cardy formula. The new massive gravity is expected to be unitary and renormalizable. It must, therefore, be interesting to obtain these results through a more direct use of the field theory of the new massive theory.


[1] S. W. Hawking, Nature **248,** 30 (1974).

[2] S. W. Hawking, Commun. Math. Phys. **43,** 199 (1975).

[3] J. M. Bardeen, B. Carter and S. W. Hawking, Commun. Math. Phys. **31,** 161 (1973).

[4] R. G. Cai and S. P. Kim, JHEP **02,** 050 (2005) [arXiv:hep-th/0501055].

[5] D. V. Fursaev, Phys. Rev. D **51,** R5352 (1995) [arXiv:hep-th/9412161].
R. B. Mann and S. N. Solodukhin, Nucl. Phys. B **523,** 293 (1998) [arXiv:hep-th/9709064].

[6] R. K. Kaul and P. Majumdar, Phys. Rev. Lett. **84,** 5255 (2000) [arXiv:gr-qc/0002040].

[7] S. Carlip, Class. Quantum Grav. **17,** 4175 (2000) [arXiv:gr-qc/0005017].
M. R. Setare, Eur. Phys. J. C **33,** (2004) [arXiv:hep-th/0309134].

[8] R. Banerjee and B. R. Majhi, JHEP **0806,** 095 (2008), [arXiv:0805.2220 [hep-th]].

[9] K. A. Meissner, Class. Quantum. Grav. **21,** 5245 (2004) [arXiv:gr-qc/0407052].

[10] S. J. Modak, Phys. Rev. B **671,** 167 (2009).

[11] M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85,** 5042 (2000) [arXiv:hep-th/9907001].
M. K. Parikh, Int. J. Mod. Phys. D **13,** 2351 (2004) [arXiv:hep-th/0405160].

[12] R. Kerner and R. B. Mann, Phys. Rev. D **73,** 104010 (2006) [arXiv:gr-qc/0603019].

[13] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102,** 201301 (2009).

[14] M. Nakasone and I. Oda, [arXiv:0902.3531 [hep-th]].

[15] I. Oda, [arXiv:0904.2833 [hep-th]].

[16] S. Deser, R. Jackiw and G.'t Hooft, Annals Phys. **152**, 220 (1984); J. D. Brown, "Lower Dimensional Gravity", S. Deser and P. O. Mazur , Class. Quant. Grav. **2,** L 51 (1985); C. Martinez, C. Teitelboim and J. Zanelli, Phys. Rev. D **61,** 104013 (2000).

[17] J. Oliva, D. Tempo and R. Troncoso, JHEP **0907,** 011 (2009).

[18] M. Akbar, [arXiv:hep-th/0702029].
M. Akbar and A. Siddiqui, Phys. Lett. B **656,** 217 (2007).

[19] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69,** 1849 (1992).

[20] R. Banerjee and B. R. Majhi, Phys. Lett. B **674,** 218 (2009), [arXiv:0808.3688 [hep-th].

[21] R. Banerjee and S. K. Modak, JHEP **0911,** 073 (2009), [arXiv:0908.2346 [hep-th]].

[22] R. Banerjee.and S. K. Modak, JHEP **0905,** 063 (2009), [arXiv:0903.3321 [hep-th]].

[23] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D **48,** 1506 (1993).

[24] G. Giribet, J. Oliva, D. Tempo and R. Troncoso, Phys. Rev. D **80,** 124046 (2009), [arXiv:0909.2564 [hep-th].

[25] R. M. Wald, Phys. Rev. D **48,** 3427 (2009), [arXiv:gr-qc/9307038].

[26] R. Aros, D. E. Diaz and A. Montecinos, JHEP 07 (2010) 012, [arXiv:1003.1083 [hep-th]].

[27] F. Caravelli, L, Modesto [arXiv:1001.4364 [gr-qc]].

[28] S. Carlip, [arXiv:0807.4192 [gr-qc]].

[29] T. Jacobson, G. King and R. C. Myers, Phys. Rev. D **49**, 6587 (1994).